

Tests for nonergodicity of denumerable continuous time Markov processes[☆]

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Abstract

We provide nonergodicity criteria for denumerable continuous time Markov processes in terms of test functions that satisfy Kaplan's condition, which resolves an open problem given by [B.D. Choi, B. Kim, nonergodicity criteria for denumerable continuous time Markov processes, and Operations Research Letters 32 (2004) 575–580]. We give two examples where the nonergodicity criteria are used.

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1. Introduction

Ergodicity and nonergodicity criteria for denumerable Markov processes have been studied for several decades. Foster [1] and Pakes [2] gave ergodicity criteria for denumerable discrete time Markov processes (DTMPs). Reuter [3] and Tweedie [4,5] studied ergodicity criteria for denumerable continuous time Markov processes (CTMPs). The paper of Anderson [6] is also a good article on ergodicity criteria for denumerable CTMPs. Kaplan [7] and Sennott et al. [8,9] gave nonergodicity criteria for denumerable DTMPs. Several ergodicity and nonergodicity criteria for denumerable DTMPs are found in Fayolle, Malyshev and Menshikov [10]. Choi and Kim [11] studied nonergodicity criteria for denumerable CTMPs.

To use ergodicity and nonergodicity criteria with test functions is almost standard in the stability analysis of queueing models in recent works. For examples, Falin [12] and He, Li and Zhao [13] studied the stability of retrial queueing systems by using ergodicity and nonergodicity criteria with test functions. For further examples, see Diamond and Alfa [14], Falin [15] and Hanschke [16].

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In this paper, we provide nonergodicity criteria for denumerable CTMPs in terms of test functions satisfying Kaplan's condition. Kaplan's condition was first introduced by Kaplan [7] for DTMPs, and it was restated by Sennott [9] in a more general form as follows: Consider a DTMP with state space E and one-step transition probabilities p_{ij} , $i, j \in E$. A function $f : E \rightarrow [0, \infty)$ is said to satisfy Kaplan's condition if

$$\inf_{c \leq z < 1, i \in E} \frac{\sum_{j \in E} p_{ij} z^{f(j)} - z^{f(i)}}{z - 1} > -\infty \quad \text{for some } c \in [0, 1). \quad (1)$$

Kaplan [7] gave a nonergodicity criterion for DTMPs and Sennott [9] extended Kaplan's result and gave the following criterion:

A DTMP is not ergodic if there exists a function $f : E \rightarrow [0, \infty)$ that satisfies

Kaplan's condition (1);

$\sum_{j \in E} p_{ij} f(j) < \infty$ *for all* $i \in E$;

$\sum_{j \in E} p_{ij} f(j) \geq f(i)$ *for all* $i \in E$;

$\sum_{j \in E} p_{ij} f(j) > f(i)$ *for some* $i \in E$.

In fact, the condition " $\sum_{j \in E} p_{ij} f(j) < \infty$ for all $i \in E$ " is redundant as we see in Section 2. Sennott [9] also proved that Kaplan's condition is weaker than

$$\sup_{i \in E} \sum_{j \in E} p_{ij} (f(i) - f(j))^+ < \infty, \quad (2)$$

where $(f(i) - f(j))^+ \equiv \max\{f(i) - f(j), 0\}$. Because the condition (2) can be checked more easily than (1) in general, it is suggested that one check (2) prior to (1) in practical applications. As a continuous time analogue of the Sennott's result, Choi and Kim [11] gave the following nonergodicity criterion for CTMPs:

A regular CTMP with denumerable state space E and Q -matrix $(q_{ij})_{i,j \in E}$ is not ergodic if there exists a nonconstant function $f : E \rightarrow [0, \infty)$ that satisfies

$$\begin{aligned} \sup_{i \in E} \sum_{j \in E} q_{ij} (f(i) - f(j))^+ &< \infty; \\ \sum_{j \in E} q_{ij} f(j) &\geq 0 \quad \text{for all } i \in E. \end{aligned} \quad (3)$$

Choi and Kim [11] also introduced a continuous time version of Kaplan's condition (1) as follows:

$$\inf_{c \leq z < 1, i \in E} \frac{\sum_{j \in E} q_{ij} z^{f(j)} - z^{f(i)}}{z - 1} > -\infty \quad \text{for some } c \in [0, 1). \quad (4)$$

The condition (4) is also called Kaplan's condition when a CTMP is considered. Choi and Kim [11] proved that Kaplan's condition (4) for CTMPs is weaker than (3), which is a continuous time analogue of Sennott's result that Kaplan's condition (1) for DTMPs is weaker than (2). However, it is not known whether the condition (3) can be replaced with Kaplan's condition (4) for their nonergodicity criteria. This paper gives a positive answer to the problem.

The remainder of the paper is organized as follows: In Section 2, we provide nonergodicity criteria for denumerable CTMPs in terms of test functions that satisfy Kaplan's condition, which resolves an open problem given by Choi and Kim [11]. In Section 3, we give two examples where the nonergodicity criteria are used. In the Appendix, we summarize some criteria for the ergodicity and nonergodicity of denumerable Markov processes found in the literature.

2. Nonergodicity criteria for denumerable CTMPs

Nonergodicity criteria for CTMPs are given in the following theorem.

Theorem 1. Consider a regular CTMP with denumerable state space E and a conservative Q -matrix $(q_{ij})_{i,j \in E}$. Let $f : E \rightarrow [0, \infty)$ be a function that satisfies Kaplan's condition (4). The CTMP is not ergodic if any of the following holds:

(a) The function f is not constant and

$$\sum_{j \in E} q_{ij} f(j) \geq 0 \quad \text{for all } i \in E.$$

(b) $\sum_{j \in E} q_{ij} f(j) \geq 0$ for all $i \in E$, and

$$\sum_{j \in E} q_{ij} f(j) > 0 \quad \text{for some } i \in E.$$

(c) There exists a nonempty subset B of E satisfying

$$\begin{aligned} \sum_{j \in E} q_{ij} f(j) &\geq 0 \quad \text{for all } i \in E - B, \text{ and} \\ f(i) &> \sup_{j \in B} f(j) \quad \text{for some } i \in E - B. \end{aligned}$$

The following lemma is used in the proof of Theorem 1.

Lemma 1. Let $(q_{ij})_{i,j \in E}$ be a conservative Q -matrix. Then, for any $f : E \rightarrow [0, \infty)$ and $i \in E$,

$$\limsup_{z \rightarrow 1-} \sum_{j \in E} \frac{q_{ij} z^{f(j)}}{1-z} \leq - \sum_{j \in E} q_{ij} f(j).$$

Proof. Let $i, j \in E$ and $z \in (0, 1)$. Then

$$\begin{aligned} \frac{z^{f(j)} - z^{f(i)}}{1-z} &= z^{f(j)} \frac{1 - z^{f(i)-f(j)}}{1-z} \\ &= z^{f(j)} (f(i) - f(j)) z_1^{f(i)-f(j)-1} \end{aligned}$$

for some $z_1 \in (z, 1)$, where the mean value theorem is used in the last equality. Hence, for $\frac{1}{2} \leq z < 1$,

$$0 \leq \frac{z^{f(j)} - z^{f(i)}}{1-z} \leq 2(f(i) - f(j)) \quad \text{if } f(j) \leq f(i). \quad (5)$$

Let $E^+ \equiv \{j \in E : f(j) > f(i)\}$ and $E^- \equiv \{j \in E : f(j) < f(i)\}$. By (5) and the Lebesgue dominated convergence theorem,

$$\lim_{z \rightarrow 1-} \sum_{j \in E^-} q_{ij} \frac{z^{f(j)} - z^{f(i)}}{1-z} = \sum_{j \in E^-} q_{ij} (f(i) - f(j)). \quad (6)$$

By Fatou's lemma,

$$\liminf_{z \rightarrow 1-} \sum_{j \in E^+} q_{ij} \frac{z^{f(i)} - z^{f(j)}}{1-z} \geq \sum_{j \in E^+} q_{ij} (f(j) - f(i)). \quad (7)$$

By (6) and (7),

$$\begin{aligned} \limsup_{z \rightarrow 1-} \sum_{j \in E} \frac{q_{ij} z^{f(j)}}{1-z} &= \limsup_{z \rightarrow 1-} \sum_{j \in E^+} q_{ij} \frac{z^{f(j)} - z^{f(i)}}{1-z} + \lim_{z \rightarrow 1-} \sum_{j \in E^-} q_{ij} \frac{z^{f(j)} - z^{f(i)}}{1-z} \\ &\leq - \sum_{j \in E^+} q_{ij} (f(j) - f(i)) - \sum_{j \in E^-} q_{ij} (f(j) - f(i)) \\ &= - \sum_{j \in E} q_{ij} f(j). \quad \square \end{aligned}$$

Proof of Theorem 1. Let $\{X(t) : t \geq 0\}$ be a regular CTMP with denumerable state space E and a conservative Q -matrix $(q_{ij})_{i,j \in E}$. Suppose that $f : E \rightarrow [0, \infty)$ satisfies (4).

First, we prove that the CTMP is not ergodic when (a) holds. Suppose that (a) holds but the CTMP is ergodic. Choose i_0 and i_1 in E such that

$$f(i_0) > f(i_1). \quad (8)$$

Choose an increasing sequence E_1, E_2, E_3, \dots of finite subsets of E such that $\{i_0, i_1\} \subset E_1$ and $\bigcup_{n=1}^{\infty} E_n = E$. For a fixed $z \in (0, 1)$, let

$$g_n(t) \equiv \mathbb{E} \left[z^{f(X(t \wedge \tau(i_1)))} \mathbf{1}_{\{\tau(E_n^c) > t \wedge \tau(i_1)\}} | X(0) = i_0 \right],$$

where

$$\tau(i_1) \equiv \inf\{t \geq 0 : X(t) = i_1\},$$

$$\tau(E_n^c) \equiv \inf\{t \geq 0 : X(t) \notin E_n\}$$

and $a \wedge b$ denotes $\min\{a, b\}$. For $t \geq 0$ and $h > 0$,

$$\begin{aligned} g_n(t+h) - g_n(t) &= \mathbb{E} \left[z^{f(X((t+h) \wedge \tau(i_1)))} \mathbf{1}_{\{\tau(E_n^c) > (t+h) \wedge \tau(i_1)\}} | X(0) = i_0 \right] \\ &\quad - \mathbb{E} \left[z^{f(X(t \wedge \tau(i_1)))} \mathbf{1}_{\{\tau(E_n^c) > t \wedge \tau(i_1)\}} | X(0) = i_0 \right] \\ &= \mathbb{E} \left[\left(z^{f(X((t+h) \wedge \tau(i_1)))} \mathbf{1}_{\{\tau(E_n^c) > (t+h) \wedge \tau(i_1)\}} - z^{f(X(t))} \mathbf{1}_{\{t < \tau(E_n^c) \wedge \tau(i_1)\}} \right) | X(0) = i_0 \right] \\ &= \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) | X(0) = i_0) \\ &\quad \times \mathbb{E} \left[z^{f(X(h \wedge \tau(i_1)))} \mathbf{1}_{\{\tau(E_n^c) > h \wedge \tau(i_1)\}} - z^{f(i)} | X(0) = i \right] \\ &= \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) | X(0) = i_0) \\ &\quad \times \left(\sum_{j \in E_n} \mathbb{P}(X(h \wedge \tau(i_1)) = j, \tau(E_n^c) > h \wedge \tau(i_1) | X(0) = i) z^{f(j)} - z^{f(i)} \right) \\ &= \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) | X(0) = i_0) \\ &\quad \times \left(\sum_{j \in E_n - \{i\}} (q_{ij}h + o(h)) z^{f(j)} + (1 + q_{ii}h + o(h)) z^{f(i)} - z^{f(i)} \right) \\ &= \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) | X(0) = i_0) \sum_{j \in E_n} (q_{ij}h + o(h)) z^{f(j)}. \end{aligned}$$

Since E_n is finite,

$$\begin{aligned} \frac{d^+}{dt} g_n(t) &\equiv \lim_{h \rightarrow 0^+} \frac{g_n(t+h) - g_n(t)}{h} \\ &= \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) | X(0) = i_0) \sum_{j \in E_n} q_{ij} z^{f(j)}. \end{aligned}$$

Hence, by a version of the fundamental theorem of calculus,

$$g_n(t) - g_n(0) = \int_0^t \sum_{i \in E_n} \mathbb{P}(X(s) = i, s < \tau(E_n^c) \wedge \tau(i_1) | X(0) = i_0) \sum_{j \in E_n} q_{ij} z^{f(j)} ds.$$

Hence

$$\begin{aligned}
z^{f(i_1)} \mathbb{P}(\tau(E_n^c) > \tau(i_1) \mid X(0) = i_0) - z^{f(i_0)} &= \lim_{t \rightarrow \infty} (g_n(t) - g_n(0)) \\
&= \int_0^\infty \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) \mid X(0) = i_0) \sum_{j \in E_n} q_{ij} z^{f(j)} dt \\
&\leq \int_0^\infty \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) \mid X(0) = i_0) \sum_{j \in E} q_{ij} z^{f(j)} dt.
\end{aligned} \tag{9}$$

Let $C_1 \equiv \sup_{i \in E} \sum_{j \in E} q_{ij} z^{f(j)}$, which is finite by (4). Then, by the monotone convergence theorem,

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \int_0^\infty \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) \mid X(0) = i_0) \left(C_1 - \sum_{j \in E} q_{ij} z^{f(j)} \right) dt \\
&= \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) \mid X(0) = i_0) \left(C_1 - \sum_{j \in E} q_{ij} z^{f(j)} \right) dt.
\end{aligned} \tag{10}$$

By the monotone convergence theorem, again,

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \int_0^\infty \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) \mid X(0) = i_0) dt \\
&= \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) \mid X(0) = i_0) dt.
\end{aligned} \tag{11}$$

Since we have assumed that the CTMP is ergodic,

$$\int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) \mid X(0) = i_0) dt = \mathbb{E}[\tau(i_1) \mid X(0) = i_0] < \infty. \tag{12}$$

Hence, by (10) and (12),

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \int_0^\infty \sum_{i \in E_n} \mathbb{P}(X(t) = i, t < \tau(E_n^c) \wedge \tau(i_1) \mid X(0) = i_0) \sum_{j \in E} q_{ij} z^{f(j)} dt \\
&= \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) \mid X(0) = i_0) \sum_{j \in E} q_{ij} z^{f(j)} dt.
\end{aligned} \tag{13}$$

By (9) and (13),

$$z^{f(i_1)} - z^{f(i_0)} \leq \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) \mid X(0) = i_0) \sum_{j \in E} q_{ij} z^{f(j)} dt. \tag{14}$$

By (4), there exists $c \in (0, 1)$ such that

$$C_2 \equiv \sup_{z \in [c, 1), i \in E} \sum_{j \in E} \frac{q_{ij} z^{f(j)}}{1 - z} < \infty.$$

By Fatou's lemma,

$$\begin{aligned}
&\liminf_{z \rightarrow 1-} \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) \mid X(0) = i_0) \left(C_2 - \sum_{j \in E} \frac{q_{ij} z^{f(j)}}{1 - z} \right) dt \\
&\geq \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) \mid X(0) = i_0) \left(C_2 - \limsup_{z \rightarrow 1-} \sum_{j \in E} \frac{q_{ij} z^{f(j)}}{1 - z} \right) dt.
\end{aligned}$$

Hence, by (12) and Lemma 1,

$$\begin{aligned} \limsup_{z \rightarrow 1-} \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) | X(0) = i_0) \sum_{j \in E} \frac{q_{ij} z^{f(j)}}{1-z} dt \\ \leq \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) | X(0) = i_0) \left(- \sum_{j \in E} q_{ij} f(j) \right) dt. \end{aligned}$$

Therefore, by the assumption (a) in the theorem,

$$\limsup_{z \rightarrow 1-} \int_0^\infty \sum_{i \in E} \mathbb{P}(X(t) = i, t < \tau(i_1) | X(0) = i_0) \sum_{j \in E} \frac{q_{ij} z^{f(j)}}{1-z} dt \leq 0. \quad (15)$$

Thus, by (14) and (15),

$$f(i_0) - f(i_1) = \lim_{z \rightarrow 1-} \frac{z^{f(i_1)} - z^{f(i_0)}}{1-z} \leq 0,$$

which contradicts (8). Hence the CTMP is not ergodic.

Next, suppose that (b) holds. If f is constant, then (ii) cannot hold. Hence f is not constant. Thus (a) holds. Hence the CTMP is not ergodic.

Finally, suppose that (c) holds. Define $g : E \rightarrow [0, \infty)$ as

$$g(i) = \max \left\{ f(i), \sup_{j \in B} f(j) \right\}.$$

Then, it can be easily shown that g satisfies the Kaplan's condition (4) and the assumptions of (a). Hence the CTMP is not ergodic. \square

From Theorem 1, we obtain the following corollary, which is a modification of Theorem 1 in Sennott [9] and a generalization of Corollary 1 in Choi and Kim [11].

Corollary 1. Consider a DTMP with denumerable state space E and one-step transition probability matrix $(p_{ij})_{i,j \in E}$. Let $f : E \rightarrow [0, \infty)$ be a function satisfying Kaplan's condition (1). The DTMP is not ergodic if any of the following holds:

- (a) The function f is not constant and $\sum_{j \in E} p_{ij} f(j) \geq f(i)$ for all $i \in E$.
- (b) $\sum_{j \in E} p_{ij} f(j) \geq f(i)$ for all $i \in E$, and $\sum_{j \in E} p_{ij} f(j) > f(i)$ for some $i \in E$.
- (c) There exists a nonempty subset B of E satisfying $\sum_{j \in E} p_{ij} f(j) \geq f(i)$ for all $i \in E - B$, and $f(i) > \sup_{j \in B} f(j)$ for some $i \in E - B$.

Proof. The corollary is proved by Theorem 1 and Lemma 2 in the Appendix. \square

Remark. The nonergodicity criterion (b) in Corollary 1 was first derived by Sennott [9] under the condition that $\sum_{j \in E} p_{ij} f(j) < \infty$ for all $i \in E$. This condition is not required for any criterion in Corollary 1.

3. Examples

3.1. Example 1: A retrieval queue

Consider a queueing system with two classes of customers and c servers. (See Fig. 1.) There is a finite queue of capacity K for class-1-customers. Class- i -customers arrive according to a Poisson process with rate λ_i , $i = 1, 2$. An arriving customer begins to receive a service immediately if there is any available server at his arrival epoch.

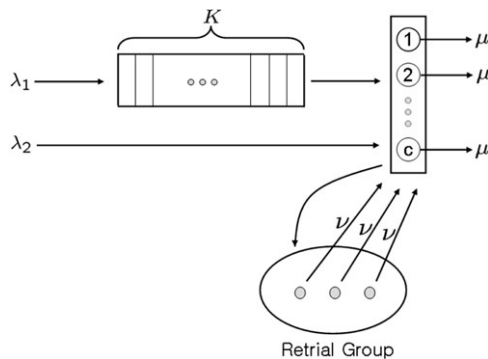


Fig. 1. A retrial queue.

Service times are independent and identically distributed with exponential distribution of mean $\frac{1}{\mu}$. An arriving class-1-customer who finds all servers busy joins the queue if the number of customers in the queue, excluding himself, is less than K . Arriving class-1-customers who find all servers busy and who find K customers in the queue are blocked and depart the system without services. If a server becomes idle and there are waiting class-1-customers in the queue, then one of the waiting class-1-customers begins to receive a service immediately. An arriving class-2-customer who finds all servers busy joins the retrial group. Each customer in the retrial group seeks an idle server after an exponential time with mean $\frac{1}{\nu}$, and begins to receive a service immediately if there is any idle server. Otherwise, he returns to the retrial group.

Let

$N(t)$ = the number of customers in the queue,

$S(t)$ = the number of busy servers,

$J(t) = N(t) + S(t)$,

$R(t)$ = the number of customers in the retrial group.

Then $\{(J(t), R(t)) : t \geq 0\}$ is a CTMP with state space $E = \{0, 1, \dots, c + K\} \times \{0, 1, 2, \dots\}$. The CTMP $\{(J(t), R(t)) : t \geq 0\}$ is irreducible and regular; see Proposition 2.3 on page 43 in [17]. The CTMP has a conservative Q -matrix $(q_{(i,r)(j,s)})_{(i,r)(j,s) \in E}$, where

$$q_{(i,r)(j,s)} = \begin{cases} \lambda_1 + \lambda_2 & \text{if } (j, s) = (i + 1, r), 0 \leq i \leq c - 1 \\ \lambda_1 & \text{if } (j, s) = (i + 1, r), c \leq i < c + K \\ \lambda_2 & \text{if } (j, s) = (i, r + 1), c \leq i \leq c + K \\ \min\{i, c\}\mu & \text{if } (j, s) = (i - 1, r) \\ r\nu & \text{if } (j, s) = (i + 1, r - 1), 0 \leq i \leq c - 1 \\ -(\lambda_1 + \lambda_2 + i\mu + r\nu) & \text{if } (j, s) = (i, r), i < c \\ -(\lambda_1 + \lambda_2 + c\mu) & \text{if } (j, s) = (i, r), c \leq i < c + K \\ -(\lambda_2 + c\mu) & \text{if } (j, s) = (i, r), i = c + K \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. The CTMP $\{(J(t), R(t)) : t \geq 0\}$ is ergodic if and only if

$$\frac{\lambda_2}{c\mu} \sum_{k=0}^K \left(\frac{\lambda_1}{c\mu} \right)^k < 1. \quad (16)$$

Proof. Let $x_i \equiv \mathbb{E}[\inf\{t \geq 0 : N(t) = 0\} | N(0) = i], i = 0, 1, \dots, K$. Then we have

$$c\mu x_{i-1} - (c\mu + \lambda_1)x_i + \lambda_1 x_{i+1} + 1 = 0, \quad i = 1, \dots, K, \quad (17)$$

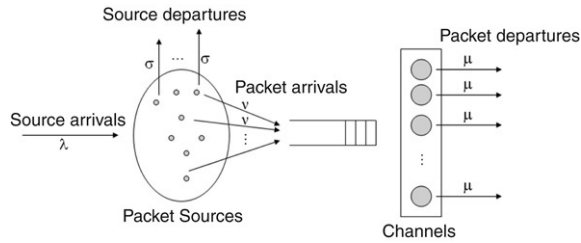


Fig. 2. A queueing system with a variable number of sources.

with $x_{K+1} \equiv x_K$. Solving the above difference equation, we have

$$x_1 = \frac{1}{\lambda_1} \sum_{i=1}^K \left(\frac{\lambda_1}{c\mu} \right)^i. \quad (18)$$

We write $x_i \equiv 0$ for $i < 0$.

Now we apply the ergodicity criterion in Theorem 6 in the Appendix to the CTMP $\{(J(t), R(t)) : t \geq 0\}$ to show that the CTMP is ergodic if (16) holds. Suppose that (16) holds and define $f : E \rightarrow [0, \infty)$ as

$$f(i, r) = \lambda_2 x_{i-c} + (1 - 2\delta) \min\{i, c\} + (1 - \delta)r, \quad (19)$$

where $\delta \in (0, \frac{1}{2})$ will be determined later. By routine calculations with (17) and (18), we obtain

$$\sum_{(j,s) \in E} q_{(i,r)(j,s)} f(j, s) = \begin{cases} (\lambda_1 + \lambda_2 - i\mu)(1 - 2\delta) - r\nu\delta, & \text{if } 0 \leq i \leq c - 1, \\ \lambda_2 \sum_{k=0}^K \left(\frac{\lambda_1}{c\mu} \right)^k - c\mu + (2c\mu - \lambda_2)\delta & \text{if } i = c, \\ -\lambda_2\delta & \text{if } c + 1 \leq i \leq c + K. \end{cases} \quad (20)$$

By (16), we can choose $\delta \in (0, \frac{1}{2})$ such that $\lambda_2 \sum_{k=0}^K \left(\frac{\lambda_1}{c\mu} \right)^k - c\mu + (2c\mu - \lambda_2)\delta < 0$. The right hand side of (20) is less than or equal to $-\epsilon$ except for finitely many (i, r) , where $\epsilon \equiv \min\{-\lambda_2 \sum_{k=0}^K \left(\frac{\lambda_1}{c\mu} \right)^k + c\mu - (2c\mu - \lambda_2)\delta, \lambda_2\delta\}$ is positive. Hence there is a finite subset B of E such that (i) of Theorem 6 holds. The conditions (ii) and (iii) of Theorem 6 can be easily checked. Thus the CTMP $\{(J(t), R(t)) : t \geq 0\}$ is ergodic if (16) holds.

Next we apply the nonergodicity criterion in Theorem 1 to show that the CTMP $\{(J(t), R(t)) : t \geq 0\}$ is not ergodic if (16) does not hold. Define $f : E \rightarrow [0, \infty)$ as

$$f(i, r) = \lambda_2 x_{i-c} + \min\{i, c\} + r.$$

We note that the function f is obtained by setting $\delta = 0$ in (19). By setting $\delta = 0$ in (20), we have

$$\sum_{(j,s) \in E} q_{(i,r)(j,s)} f(j, s) \geq 0$$

for all $(i, r) \in E$, unless (16) holds. It can be easily checked that the condition (3) holds, which is stronger than Kaplan's condition (4). Therefore, by the nonergodicity criterion (a) of Theorem 1, the CTMP $\{(J(t), R(t)) : t \geq 0\}$ is not ergodic if (16) does not hold. \square

Remark. If the transition rate for a CTMP is bounded, then we can study the ergodicity and nonergodicity of the CTMP by using ergodicity and nonergodicity criteria for DTMPs through Lemma 3 in the Appendix. However, because the transition rates of the CTMP $\{(J(t) < R(t)) : t \geq 0\}$ is not bounded, ergodicity and nonergodicity criteria for DTMPs cannot be applied directly to the example in this section.

3.2. Example 2: A queueing system with a variable number of sources

Consider a queueing system with a variable number of packet sources. (See Fig. 2.) Packet sources arrive at the system according to a Poisson process with intensity λ . The lifetimes of the sources are assumed to be independent

and exponentially distributed with mean $\frac{1}{\sigma}$. Each source in the system generates packets with intensity ν . There are c channels, and the service (transmission) times of packets are independent and exponentially distributed with mean $\frac{1}{\mu}$. A packet that finds all channels busy waits in the queue until a channel becomes available.

Let $N_1(t)$ ($N_2(t)$, respectively) be the number of sources (packets, respectively) in the systems at time t . Then $\{(N_1(t), N_2(t)) : t \geq 0\}$ is a CTMP with state space $E = \mathbb{Z}^+ \times \mathbb{Z}^+$, where $\mathbb{Z}^+ \equiv \{0, 1, 2, \dots\}$. The CTMP $\{(N_1(t), N_2(t)) : t \geq 0\}$ is irreducible and regular; see Proposition 2.3 on page 43 in [17]. The CTMP has a conservative Q -matrix $(q_{(i,j)(k,l)})_{(i,j)(k,l) \in E}$, where

$$q_{(i,j)(k,l)} = \begin{cases} \lambda & \text{if } (k, l) = (i + 1, j) \\ i\sigma & \text{if } i \geq 1 \text{ and } (k, l) = (i - 1, j) \\ i\nu & \text{if } (k, l) = (i, j + 1) \\ \min\{j, c\} \mu & \text{if } j \geq 1 \text{ and } (k, l) = (i, j - 1) \\ -(\lambda + i\sigma + i\nu + \min\{j, c\} \mu) & \text{if } (k, l) = (i, j) \\ 0 & \text{otherwise.} \end{cases}$$

Now we study the ergodicity condition for the CTMP $\{(N_1(t), N_2(t)) : t \geq 0\}$. Packet sources arrive at the rate λ and the mean number of packets generated by a source during its life time is $\frac{\nu}{\sigma}$. Thus, the long run arrival rate of packets is $\lambda \frac{\nu}{\sigma}$. On the other hand, the departure rate of packets is $c\mu$ when all servers are working. Therefore one may conjecture that the CTMP is ergodic if and only if $\lambda \frac{\nu}{\sigma} < c\mu$. We confirm this in the following theorem by the nonergodicity criterion (a) in Theorem 1, and a well known ergodicity criterion that is given in the Appendix for the completeness of the paper.

Theorem 3. *The CTMP $\{(N_1(t), N_2(t)) : t \geq 0\}$ is ergodic if and only if*

$$\frac{\lambda\nu}{c\mu\sigma} < 1. \quad (21)$$

Proof. First, we apply an ergodicity criterion in Theorem 6 in the Appendix to the CTMP $\{(N_1(t), N_2(t)) : t \geq 0\}$ to show that the CTMP is ergodic if (21) holds.

Suppose that (21) holds. Choose $\delta > 0$ such that

$$\lambda(\nu + \delta) < c\mu\sigma \quad (22)$$

and define $f : E \rightarrow [0, \infty)$ as

$$f(i, j) = (\nu + \delta)i + \sigma j. \quad (23)$$

Then

$$\sum_{(k,l) \in E} q_{(i,j)(k,l)} f(k, l) = \lambda(\nu + \delta) - i\sigma\delta - \min\{j, c\}\mu\sigma. \quad (24)$$

Let

$$\epsilon \equiv c\mu\sigma - \lambda(\nu + \delta) > 0;$$

$$B \equiv \{(i, j) \in E : i\delta < c\mu, j < c\}.$$

Then (24) implies that

$$\sum_{(k,l) \in E} q_{(i,j)(k,l)} f(k, l) \leq -\epsilon \quad \text{if } (i, j) \in E - B.$$

Clearly, B is finite and conditions (ii) and (iii) in Theorem 6 hold. Therefore the CTMP $\{(N_1(t), N_2(t)) : t \geq 0\}$ is ergodic by Theorem 6.

Next, by using the nonergodicity criterion (a) in Theorem 1, we show that the CTMP $\{(N_1(t), N_2(t)) : t \geq 0\}$ is not ergodic if (21) does not hold.

Suppose that (21) does not hold. Define $f : E \rightarrow [0, \infty)$ by

$$f(i, j) = \nu i + \sigma j.$$

We notice that the function f is obtained by setting $\delta = 0$ in (23). By setting $\delta = 0$ in (24), we have

$$\sum_{(k,l) \in E} q_{(i,j)(k,l)} f(k,l) = \lambda v - \min\{j, c\} \mu \sigma \geq \lambda v - c \mu \sigma \geq 0,$$

for all $(i, j) \in E$.

Now we prove that f satisfies Kaplan's condition (4). For $(i, j) \in E$ and $z \in (0, 1)$,

$$\begin{aligned} \sum_{(k,l) \in E} \frac{q_{(i,j)(k,l)} z^{f(k,l)}}{1-z} &= \frac{z^{vi+\sigma j} \{ \lambda(z^v - 1) + i\sigma(z^{-v} - 1) + iv(z^\sigma - 1) + \min\{j, c\} \mu(z^{-\sigma} - 1) \}}{1-z} \\ &\leq \frac{z^{vi} \{ i\sigma(z^{-v} - 1) + iv(z^\sigma - 1) \}}{1-z} + \frac{c\mu(z^{-\sigma} - 1)}{1-z} \\ &= iz^{v(i-1)} \frac{vz^{v+\sigma} - (v+\sigma)z^v + \sigma}{1-z} + \frac{c\mu}{z^\sigma} \frac{1-z^\sigma}{1-z}. \end{aligned} \quad (25)$$

By the mean value theorem, $\frac{1-z^\sigma}{1-z} = \sigma z_1^{\sigma-1}$ for some $z_1 \in (z, 1)$. Thus $\frac{1-z^\sigma}{1-z} \leq \frac{\sigma}{z}$. Therefore

$$\sup_{\frac{1}{2} \leq z < 1} \frac{c\mu}{z^\sigma} \frac{1-z^\sigma}{1-z} \leq \sup_{\frac{1}{2} \leq z < 1} \frac{c\mu\sigma}{z^{\sigma+1}} = 2^{\sigma+1} c\mu\sigma < \infty. \quad (26)$$

By the mean value theorem again,

$$\frac{vz^{v+\sigma} - (v+\sigma)z^v + \sigma}{1-z} = v(v+\sigma) (z_2^{v-1} - z_2^{v+\sigma-1})$$

for some $z_2 \in (z, 1)$. Hence

$$\begin{aligned} iz^{v(i-1)} \frac{vz^{v+\sigma} - (v+\sigma)z^v + \sigma}{1-z} &= iz^{v(i-1)} v(v+\sigma) (z_2^{v-1} - z_2^{v+\sigma-1}) \\ &\leq iz_2^{v(i-1)} v(v+\sigma) (z_2^{v-1} - z_2^{v+\sigma-1}) \\ &= iv(v+\sigma) z_2^{-1} (z_2^{vi} - z_2^{vi+\sigma}). \end{aligned} \quad (27)$$

Since $h(x) \equiv x^{vi} - x^{vi+\sigma}$, $x \geq 0$, has a maximum at $x = \left(\frac{vi}{vi+\sigma}\right)^{\frac{1}{\sigma}}$, the right hand side of (27) is bounded by $iv(v+\sigma) z_2^{-1} \left(\frac{vi}{vi+\sigma}\right)^{\frac{v}{\sigma}i} \frac{\sigma}{vi+\sigma}$. Therefore

$$\begin{aligned} \sup_{\frac{1}{2} \leq z < 1, i \in \mathbb{Z}^+} iz^{v(i-1)} \frac{vz^{v+\sigma} - (v+\sigma)z^v + \sigma}{1-z} &\leq \sup_{\frac{1}{2} \leq z < 1, i \in \mathbb{Z}^+} v(v+\sigma) \frac{1}{z} \left(\frac{vi}{vi+\sigma}\right)^{\frac{v}{\sigma}i} \frac{i\sigma}{vi+\sigma} \\ &\leq v(v+\sigma) 2 \sup_{i \in \mathbb{Z}^+} \left(\frac{vi}{vi+\sigma}\right)^{\frac{v}{\sigma}i} \frac{\sigma}{v}. \end{aligned}$$

Since $\lim_{i \rightarrow \infty} \left(\frac{vi}{vi+\sigma}\right)^{\frac{v}{\sigma}i} = e^{-1} < \infty$, $\sup_{i \in \mathbb{Z}^+} \left(\frac{vi}{vi+\sigma}\right)^{\frac{v}{\sigma}i} < \infty$. Hence

$$\sup_{\frac{1}{2} \leq z < 1, i \in \mathbb{Z}^+} iz^{v(i-1)} \frac{vz^{v+\sigma} - (v+\sigma)z^v + \sigma}{1-z} < \infty. \quad (28)$$

By (25), (26) and (28),

$$\sup_{\frac{1}{2} \leq z < 1, (i,j) \in E} \sum_{(k,l) \in E} \frac{q_{(i,j)(k,l)} z^{f(k,l)}}{1-z} < \infty.$$

Thus Kaplan's condition (4) holds and the CTMP $\{(N_1(t), N_2(t)) : t \geq 0\}$ is not ergodic by the nonergodicity criterion (a) in Theorem 1. \square

- Remark.** 1. When $\frac{\lambda\nu}{c\mu\sigma} \geq 1$, the function f does not satisfy (3). Hence we cannot apply the nonergodicity criteria in Choi and Kim [11].
2. It is not very difficult to find alternative proofs for Theorems 2 and 3. We provide the examples in this section to illustrate that ergodicity or nonergodicity for many CTMPs can be proved easily by using Theorems 1 and 6.

Appendix

We summarize some criteria for the ergodicity and nonergodicity of denumerable Markov processes found in the literature.

Theorem 4 (Foster [1], Pakes [2], Theorem 2.2.3 in [10], Ergodicity criteria for DTMPs). Consider an irreducible and aperiodic DTMP with denumerable state space E and one-step transition probability matrix $P = (p_{ij})$. Let B be a finite subset of E . Then the DTMP is ergodic if and only if there exists a function $f : E \rightarrow [0, \infty)$ and $\epsilon > 0$ satisfying

- (i) $\sum_{j \in E} p_{ij} f(j) - f(i) \leq -\epsilon, i \in E - B;$
- (ii) $\sum_{j \in E} p_{ij} f(j) < \infty, i \in B.$

Theorem 5 (Reuter [3], Tweedie [5], Theorem 6.2.3 in [6], Ergodicity criteria for CTMP). Consider an irreducible regular CTMP with denumerable state space E and Q -matrix $Q = (q_{ij})$. Let B be a finite subset of E . Then the CTMP is ergodic if and only if there exists a function $f : E \rightarrow [0, \infty)$ and $\epsilon > 0$ satisfying

- (i) $\sum_{j \in E} q_{ij} f(j) \leq -\epsilon, i \in E - B;$
- (ii) $\sum_{j \in E} q_{ij} f(j) < \infty, i \in B.$

Theorem 6 (Tweedie [4], Ergodicity criteria for CTMP). Consider an irreducible CTMP with denumerable state space E and Q -matrix $(q_{ij})_{i,j \in E}$. The CTMP is regular and ergodic if there exists a function $f : E \rightarrow [0, \infty)$, a finite subset B of E and $\epsilon > 0$ such that

- (i) $\sum_{j \in E} q_{ij} f(j) \leq -\epsilon$ for all $i \in E - B;$
- (ii) $\sum_{j \in E} q_{ij} f(j) < \infty$ for all $i \in B;$
- (iii) the set $\{i \in E : f(i) \leq L\}$ is finite for any $L < \infty$.

It is well known that an irreducible DTMP with one-step transition probability matrix P is positive recurrent if and only if there is a probability vector π such that $\pi P = \pi$. As a continuous time analogue, an irreducible and regular CTMP with Q -matrix Q is positive recurrent if and only if there is a probability vector p such that $pQ = 0$. The following lemmas are immediate from the above facts.

Lemma 2. An irreducible DTMP with one-step transition probability matrix P is positive recurrent if and only if the CTMP with the Q -matrix $Q \equiv P - I$ is ergodic.

Lemma 3. Consider a CTMP with denumerable state space E and a conservative Q -matrix $(q_{ij})_{i,j \in E}$. Suppose that $\sup_{i \in E} |q_{ii}| \leq \gamma < \infty$. Then the CTMP is ergodic if and only if a DTMP with one-step transition probabilities $p_{ij} \equiv \delta_{ij} + \frac{1}{\gamma} q_{ij}, i, j \in E$, is positive recurrent, where δ_{ij} denotes 1 if $i = j$, and 0 otherwise.

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